The Well Structured Problem for Presburger Counter Machines

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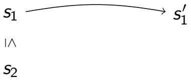
Well Structured Transition Systems [Finkel 87]

$(X, \to, \leq)$ is a well structured transition system iff

- $(X, \leq)$ is a well quasi order.
- $(X, \to, \leq)$ is monotone.
  - if there is a transition $s_1 \to s_1'$ and $s_2 \geq s_1$:



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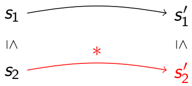

  - then there must exist $s_2' \geq s_1'$ such that $s_2 \xrightarrow{*} s_2'$.

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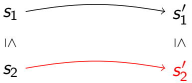

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Well Structured Transition Systems

Examples:

- Petri Nets
- Vector addition systems with states (VASS)
- VASS with reset/transfer/affine-ω extensions
- Finite state automata
- Lossy fifo systems and variants with time, data and priority
- ...

(Finkel, Schnoebelen, 2001)

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Coverability is decidable for WSTS.

- backward algorithm on upward closed sets.
  (Abdulla, Cerans et al., 1996)
- forward algorithm on downward closed sets.
  (Blondin, Finkel et al., 2017)

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Minsky Machines

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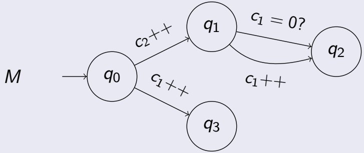

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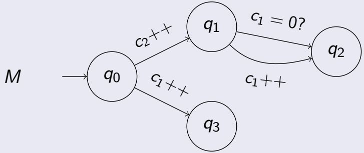

Observe that M is (strongly) monotone.

Let us consider new problems about WSTS:

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First order formulae over $(\mathbb{N}, +)$.

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- $lessthan(x, y) : \exists z(x + z = y)$
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- $\phi(x, y) : y = x + x + x + 3$

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Theorem (Presburger, 1929)

Presburger arithmetic is decidable.

Presburger Counter Machines

d-dimensional PCM $M = (Q, \rightarrow, \Phi)$:

- d counters.
- Q : set of control-states.
- Φ: set of Presburger formulae having $2d$ free variables.
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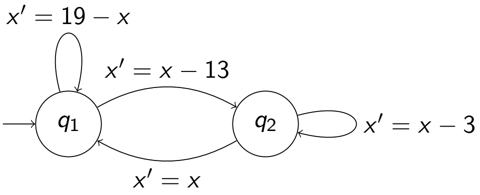

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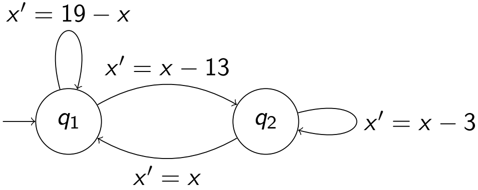

Sample run:
$(q_1, 0) \rightarrow (q_1, 19) \rightarrow (q_2, 6) \rightarrow (q_2, 3) \rightarrow (q_1, 3)$

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- Positive AVASS (self-modified nets): AVASS where $A \in \mathbb{N}^{d\times d}$.
- Totally Positive AVASS: Positive AVASS where $b \in \mathbb{N}^d$.

Well Structured Problem

Our Results:

	Well Structured Problem	Strong Well Structured Problem
PCM	U	D
Functional 1-PCM	U	D
2-AVASS	U	D
2-Minsky machines	U	D
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strong well-structuredness is Presburger expressible

Undecidability of WSP for 1-PCM

- Reduction from Minsky machine reachability.
- Given Minsky machine $M = (Q, \rightarrow, q_0)$, convert to 1-PCM N such that:

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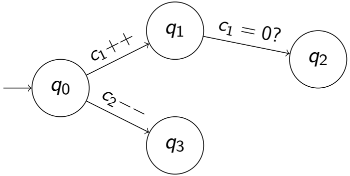
Consider (q, n) and (q, m) equivalent if $(\nu_2(n) = \nu_2(m)) \wedge (\nu_3(n) = \nu_3(m))$.

Given $M = (Q, \rightarrow, q_0)$:



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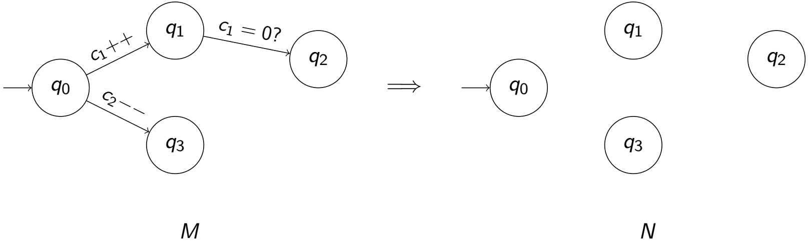

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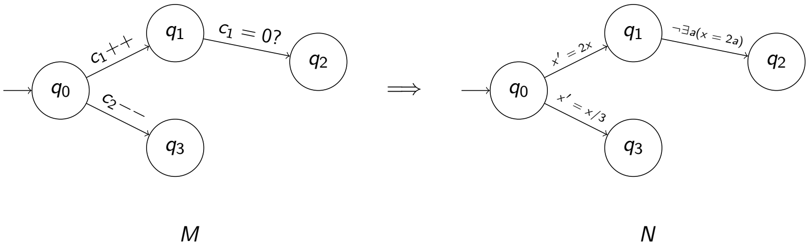

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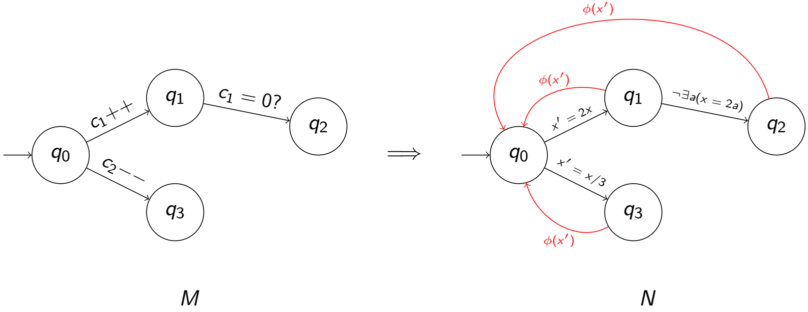

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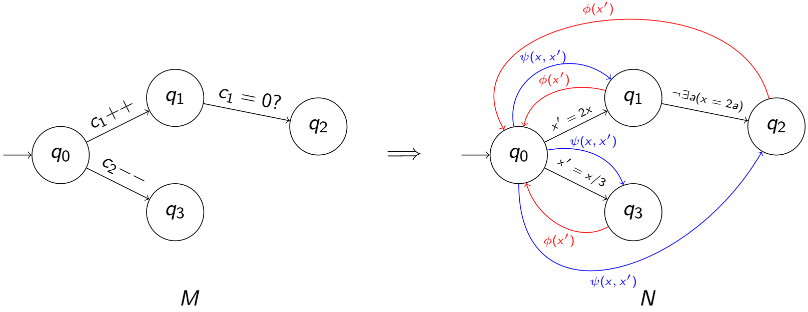

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$\psi(x, x') = (x' = x = 0)$

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Well Structured Problem

Theorem

WSP is undecidable for:

- *Functional* 1*-dim PCMs.*
- *2 counter Minsky machines.*
- *2 AVASS.*

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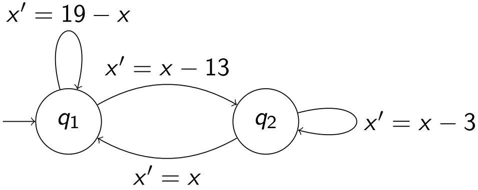

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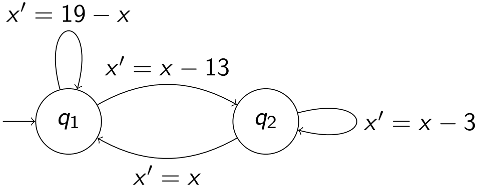

1-AVASS have the following properties:

- Reachability and coverability are decidable.
  (Finkel, Goller, Hasse, 2013)
- $Pre^*(q, n)$ is computable.
- WSP is decidable.

Pre^* algorithm

Given 1-AVASS $M = (Q, \rightarrow)$, configuration (q_f, n_f). Compute $Pre^*(q_f, n_f)$:

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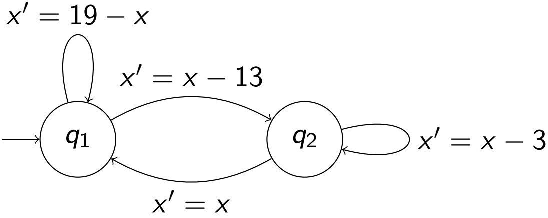
Idea:

- Maintain Presburger formula for each state q storing $\{n : (q, n) \in Pre^*(q_f, n_f)\}$.
- Iteratively backtrack from each transition and back-accelerate simple cycles.

Example: Compute $Pre^*(q_1, 19)$:



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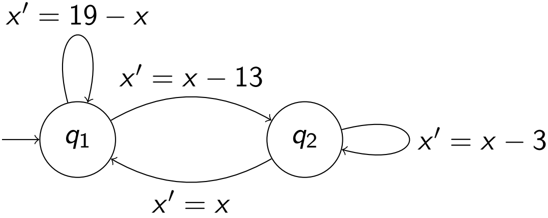

ϕ_1:$n = 19$

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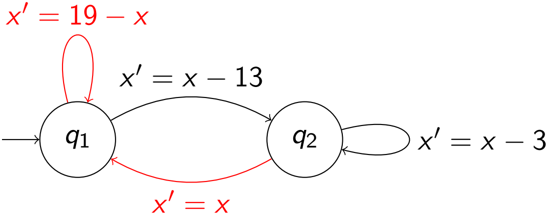

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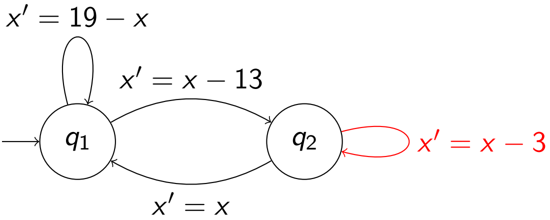

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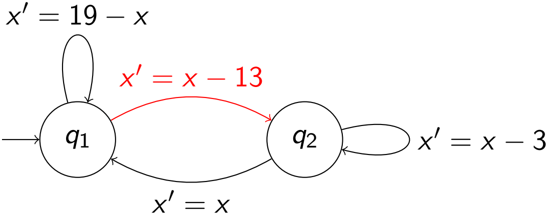

ϕ_1:$n \in \{19, 0\} \vee$
$(n \geq 32 \wedge n =_3 2) \vee$
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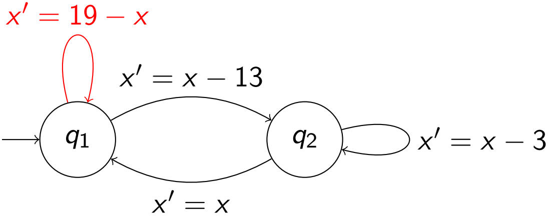

ϕ_1: $n \in \{0, 3, 6, 19\} \vee$
$(n \geq 32 \wedge n =_3 2) \vee$
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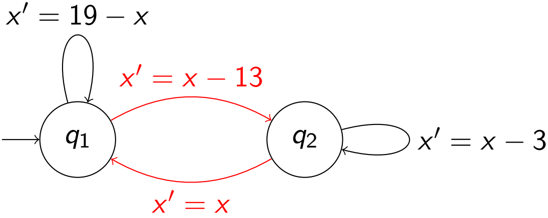

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$(n \geq 45 \wedge n =_3 0)$

ϕ_2:$(n \geq 19 \wedge n =_3 1) \vee$
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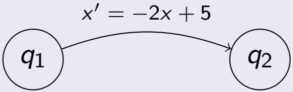

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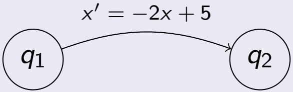

$(q_1, 0) \rightarrow (q_2, 5)$. So for WSTS, $(q_1, n) \xrightarrow{*} (q_2, m \geq 5)$.

- M is a WSTS, iff for all negative transitions $(q_1, (x' = ax + b), q_2)$, the set $\{q_1\} \times \mathbb{N}$ is a subset of $Pre^*(\uparrow(q_2, b))$.

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Further work

- Complexity analysis of the WSP on 1-AVASS.
- Complexity of the computation of Pre^* for 1-AVASS.
- Solve WSP for other models like pushdown counter machines, fifo automata, Petri net extensions.

Thank you!